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POWER CLOSURE AND THE ENGEL CONDITION*

BY

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ABSTRACT

A Lie p-algebra L is called n-power closed if, in every section of L, any sum of p^{i+n}th powers is a p^ith power ($i > 0$). It is easy to see that if L is p^n-Engel then it is n-power closed. We establish a partial converse to this statement: if L is residually nilpotent and n-power closed for some $n \geq 0$ then L is $(3p^{n+2}+1)$-Engel if $p > 2$ and $(3 \cdot 2^{n+3}+1)$-Engel if $p = 2$. In particular, then L is locally nilpotent by a theorem of Zel'manov. We deduce that a finitely generated pro-p group is a Lie group over the p-adic field if and only if its associated Lie p-algebra is n-power closed for some n. We also deduce that any associative algebra R generated by nilpotent elements satisfies an identity of the form $(x+y)^{p^n} = x^{p^n} + y^{p^n}$ for some $n \geq 1$ if and only if R satisfies the Engel condition.

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1. Introduction

The purpose of this article is to study certain additive properties of the p-map of a Lie p-algebra that turn out to be intimately related to the Engel condition. If a Lie p-algebra L satisfies the identity

$$(x+y)^{p^n} = x^{p^n} + y^{p^n}$$

we shall say that L is p^n-additive; we call L power-additive if L is p^n-additive for some $n \geq 1$. Power-additive associative algebras were studied by the first author in [R2]. There it was shown that an associative algebra R over an infinite field of characteristic $p > 0$ is power-additive if and only if R satisfies the Engel condition. This result does not extend to the class of all Lie p-algebras, however, because satisfying the Engel condition is insufficient cause to conclude that a non-associative Lie p-algebra is power-additive. Perhaps the most simple counter-example is the following.

Example 1.1: Let $L = \langle x, y \rangle$ be the free nilpotent 2-generated Lie 3-algebra of class 3. Then $(a+b)^3 = a^3 + b^3 + [a, b-a, b-a]$ for all a, b in L. Because $[x, y-x, y-x]$ is central but not nil, L cannot be power-additive.

Therefore, in order to extend this characterisation of associative algebras satisfying the Engel condition to general Lie p-algebras, we need to weaken somewhat the notion of power-additivity. Adopting the terminology originally used by Mann ([M]) and Shalev ([Sh2]) to study an analogous property in p-groups, we shall say that L is n-power closed if, in every section of L, any sum of two p^{i+n}th powers is a p^ith power ($i > 0$). It is not difficult to verify that if L is p^n-additive then it is n-power closed, but, more to the point, we have the following straightforward result.

LEMMA 1.2: *Let L be any Lie p-algebra. Then the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold for the following statements:*

1. *L is p^n-Engel.*
2. *L satisfies the identity $[x^{p^n}, y^{p^n}] = 0$.*
3. *L satisfies the identity $x^{p^{n+1}} + y^{p^{n+1}} = (x^{p^n} + y^{p^n})^p$.*
4. *L is n-power closed.*

Proof: Assume that L satisfies the p^n-Engel condition:

$$x(\operatorname{ad} y)^{p^n} = [x,_{p^n} y] = 0.$$

This is equivalent to L satisfying the identity $[x, y^{p^n}] = 0$. In particular, then L satisfies $[x^{p^n}, y^{p^n}] = 0$, and hence satisfies $x^{p^{n+1}} + y^{p^{n+1}} = (x^{p^n} + y^{p^n})^p$. By replacing x and y by x^p and y^p, respectively, it follows by induction that L satisfies the identities $x^{p^{n+i}} + y^{p^{n+i}} = (x^{p^n} + y^{p^n})^{p^i}$ for all $i > 0$. This clearly is sufficient to conclude that L is n-power closed. ∎

In fact, for a large class of Lie p-algebras the property of being n-power closed for some $n \geq 0$ turns out to be equivalent to satisfying the Engel condition.

THEOREM A: *Let L be a residually nilpotent Lie p-algebra, and suppose that L is n-power closed for some $n \geq 0$. Then L satisfies the $(3p^{n+2} + 1)$-Engel condition in the case that p is odd, and the $(3 \cdot 2^{n+3} + 1)$-Engel condition in the case that $p = 2$.*

The following is a direct consequence of Theorem A and Lemma 1.1.

COROLLARY A1: *Let L be a residually nilpotent Lie p-algebra. Then the following properties are equivalent:*

1. *L satisfies the Engel condition.*
2. *L satisfies the identity $[x^{p^n}, y^{p^n}] = 0$ for some $n \geq 0$.*
3. *L satisfies the identity $x^{p^{n+1}} + y^{p^{n+1}} = (x^{p^n} + y^{p^n})^p$ for some $n \geq 0$.*
4. *L is n-power closed for some $n \geq 0$.*

Using Zel'manov's theorem ([Z1] and [Z2]) on the local nilpotence of modular Lie algebras satisfying the Engel condition, we can then deduce that for finitely generated Lie p-algebras each of the properties listed in Corollary A1 is equivalent to nilpotence.

COROLLARY A2: *Let L be a d-generated residually nilpotent Lie p-algebra. If L is n-power closed for some $n \geq 0$ then L is nilpotent of class bounded by a fixed function of n, p, and d only.*

Proof: According to Theorem A, L satisfies the m-Engel condition for some fixed function m depending on n and p only. Therefore, by Zel'manov's theorem, L is nilpotent of class at most c, where c is the class of the relatively-free d-generated Lie p-algebra satisfying the m-Engel condition. ∎

As a further application of Theorem A we are able to eliminate the restriction on the cardinality of base field in the result from [R2] mentioned above, but at the cost of requiring the algebra to be generated by nilpotent elements. Implicit in the proof is a deep theorem of Braun [Br]: the Jacobson radical of an affine PI-algebra is nilpotent.

THEOREM B: *Let R be an associative algebra over any field of characteristic $p > 0$, and suppose that R can generated by nilpotent elements. Then the following properties are equivalent:*

1. *R satisfies the Engel condition.*
2. *R satisfies $[x^{p^n}, y^{p^n}] = 0$ for some $n \geq 0$.*
3. *R satisfies $x^{p^{n+1}} + y^{p^{n+1}} = (x^{p^n} + y^{p^n})^p$ for some $n \geq 0$.*
4. *R is power-additive.*

By Shalev's associative algebra analogue [Sh1] of Zel'manov's theorem referred to above, if R is affine then the properties listed above in Theorem B are each equivalent to R being nilpotent as a Lie algebra.

Our final application is a characterisation of p-adic analytic pro-p groups G in terms of the structure of their associated Lie p-algebras $\mathcal{L}(G)$. See Section 5 for the definition of $\mathcal{L}(G)$. A broad discussion of pro-p groups can be found in the monograph [DDMS].

THEOREM C: *Let G be a finitely generated pro-p group. Then G is p-adic analytic if and only if $\mathcal{L}(G)$ is n-power closed for some $n \geq 0$.*

Other characterisations of the p-adic analyticity of G in terms of the structure of $\mathcal{L}(G)$ were given previously by Lazard [L], Shalev [Sh3], and the present authors in [R1] and [RS]. The proof of Theorem C will use Lazard's theorem: the analyticity of G and the nilpotence of $\mathcal{L}(G)$ are equivalent conditions. Shalev gave another criterion for analyticity in [Sh2]; namely, a finitely generated pro-p group is p-adic analytic if and only if it is m-power closed for some m. Theorem C now yields the following immediate consequence.

COROLLARY C1: *Let G be a finitely generated pro-p group. Then G is m-power closed for some $m \geq 0$ if and only if $\mathcal{L}(G)$ is n-power closed for some $n \geq 0$.*

It is important to stress that the ideas we employ below follow closely those used by Shalev to prove [Sh2, Corollary], which may be viewed as a group-theoretical analogue of Theorem A. Many of the intermediate steps also have their group-theoretic counterparts. Let us also point out that the concept of a powerful Lie p-algebra, which was studied in detail by the authors in [RS], was originally inspired by the Lubotzky–Mann theory of powerful p-groups ([M] and [LM1]).

2. The finite dimensional case

Let F be a fixed base field of characteristic $p > 0$, and let $\mathcal{F}_p$ denote the class of all finite dimensional Lie p-algebras over F whose p-map acts nilpotently. By Engel's theorem, all such algebras are nilpotent. Throughout this section we shall assume that L lies in the class $\mathcal{F}_p$.

The heart of our argument is to show that the only obstruction to L satisfying a given Engel condition is the 'involvement' in L of certain canonical 2-generated Lie p-algebras. We say that a Lie p-algebra W is involved in L if W is isomorphic to a section of L. By a section of L we mean a quotient H/K, where K is an ideal of H and H is a subalgebra of L; we say that H/K is a section of ideals of L if H is an ideal of L. All subalgebras are assumed to be closed under the p-map.

Let us now define the restricted-Lie-theoretic analogue of the group wreath product $C_p \wr C_{p^n}$. Wreath products of Lie algebras are discussed in [B].

Definition 2.1: For each integer $n \geq 1$, let W_n denote the Lie p-algebra generated by x and y subject only to the following relations:

1. $a_i = [x,_i y]$ and $b_i = y^{p^i}$ for each $i \geq 0$;
2. $[a_i, a_j] = 0$ and $a_i^p = 0$ for each $i, j \geq 0$; and,
3. $a_{p^n} = b_n = 0$.

LEMMA 2.2: *The following statements hold for all n.*

1. *W_n has basis $a_0, a_1, \ldots, a_{p^n-1}, b_0, b_1, \ldots, b_{n-1}$.*
2. *$(x+y)^{p^j} = a_{p^j-1} + b_j$ for all $1 \leq j \leq n$.*
3. *W_n is not p^n-additive.*
4. *$a_{p^{n-1}-1}$ is not a p-power in W_n.*

Proof: 1. This is a straightforward consequence of the definition.

2. Let us begin with the case $j = 1$ and let λ be an indeterminate. Then

$$(x+y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x,y)$$

where $is_i(x,y)$ is defined to be the coefficient of λ^{i-1} in the expansion of $[x,_{p-1}(\lambda x + y)]$. But clearly

$$[x,_{p-1}(\lambda x + y)] = [x,_{p-1} y] = a_{p-1}.$$

So, $s_1(x,y) = a_{p-1}$ and $is_i(x,y) = 0$ for all $i \neq 1$. Thus $(x+y)^p = a_{p-1} + b_1$ as claimed. Assume now by induction that the result holds for $j-1 < n$. Then

$$\begin{aligned}(x+y)^{p^j} &= \left((x+y)^{p^{j-1}}\right)^p \\ &= (a_{p^{j-1}-1} + b_{j-1})^p \\ &= b_j + \sum_{i=1}^{p-1} s_i(a_{p^{j-1}-1}, b_{j-1}).\end{aligned}$$

But since

$$\begin{aligned}[a_{p^{j-1}-1},_{p-1}(\lambda a_{p^{j-1}-1} + b_{j-1})] &= [x,_{p^{j-1}-1} y,_{(p-1)p^{j-1}} y] \\ &= a_{p^j-1},\end{aligned}$$

the result follows as for the case $j = 1$.

3. By part (2) we have $(x+y)^{p^n} = a_{p^n-1} \neq 0$. Therefore because $x^{p^n} = y^{p^n} = 0$, W_n is not p^n-additive.

4. To the contrary, assume that $a_{p^{n-1}-1} = z^p = (a+b)^p$, where $a = \alpha_0 a_0 + \cdots + \alpha_{p^n-1} a_{p^n-1}$ and $b = \beta_0 b_0 + \cdots + \beta_{n-1} b_{n-1}$, where the α_i and β_j are scalars. Then arguing as in part (2) we find that

$$\begin{aligned}0 &= z^{p^2} \\ &= (a^p + b^p + [a,_{p-1} b])^p \\ &= b^{p^2} + [a,_{p-1} b]^p + [[a,_{p-1} b],_{p-1} b^p] \\ &= b^{p^2} + [a,_{p^2-1} b].\end{aligned}$$

Since b^{p^2} and $[a,_{p^2-1} b]$ are linearly independent, they must therefore both be zero. Now

$$0 = b^{p^2} = \beta_0^{p^2} b_0^{p^2} + \cdots + \beta_{n-1}^{p^2} b_{n-1}^{p^2} = \beta_0^{p^2} y^{p^2} + \cdots + \beta_{n-3}^{p^2} y^{p^{n-1}}$$

so that $\beta_0 = \cdots = \beta_{n-3} = 0$, and hence $b = \beta_{n-2} y^{p^{n-2}} + \beta_{n-1} y^{p^{n-1}}$. This implies that

$$a_{p^{n-1}-1} = z^p = b^p + [a,_{p-1} b] = \beta_{n-2}^p y^{p^{n-1}} + [a,_{p-1} b].$$

Again by linear independence this yields $\beta_{n-2}^p = 0$, so that $b = \beta_{n-1} y^{p^{n-1}}$. However, then

$$a_{p^{n-1}-1} = z^p = [a,_{p-1} (\beta_{n-1} y^{p^{n-1}})],$$

the latter being contained in the span of

$$\{a_{(p-1)p^{n-1}}, a_{(p-1)p^{n-1}+1}, \ldots\}.$$

Now using part (1) once more, we finally obtain the desired contradiction because $p^{n-1} - 1 < (p-1)p^{n-1}$ for all $n \geq 1$. ∎

COROLLARY 2.3: *W_{n+2} is not n-power closed ($n \geq 0$).*

Proof: From above we know that $(x+y)^{p^{n+1}} + (-y)^{p^{n+1}} = a_{p^{n+1}-1}$ is not a p-power in $W_{p^{n+2}}$. The result now follows immediately from the definition of being n-power closed. ∎

It is now clear that if L is n-power closed then it cannot involve W_{n+2}. We shall prove that this latter condition forces L to contain a large 'powerful' ideal. But first, we introduce some more notation. Let us write $\gamma_i(L)$ for the restricted closure of the ith term of the lower central series of L, and denote by L^{p^n} the ideal of L generated by elements x^{p^n}, x in L. We often abbreviate $\gamma_2(L)$ by L', and we let $\Phi(L) = L' + L^p$ represent the Frattini subalgebra of L. The centre of L will be written as $\zeta(L)$. Finally, as in [RS], we say that L (in $\mathcal{F}_p$) is powerful provided that $L' \subseteq L^p$ when p is odd, and $L' \subseteq L^4$ when $p = 2$.

The next lemma is easily deduced from Lincoln and Towers, [LT].

LEMMA 2.4: *Every maximal subalgebra of L is an ideal of codimension* 1.

We are now ready to prove Lie-theoretic analogues of the p-group results [Sh2, Lemma], [Sh2, Proposition] and [Sh2, Theorem], respectively.

LEMMA 2.5: *Let M be an ideal of L which centralises every abelian section of ideals of L on which the p-map acts trivially. If $p > 2$ then M is powerful; if $p = 2$ then the ideal M^2 of L is powerful.*

Proof: We use induction on $\dim M$, $M \neq 0$. First consider the case $p \geq 3$. The inclusion $M^p + [M, L] < M$ is proper. Take a subalgebra N satisfying

$$M^p + [M, L] \leq N < M,$$

and assume N is maximal in M. Then by the preceding lemma we see that N is a ideal of L with $\dim M/N = 1$. By the induction hypothesis N is powerful, so we may factor L by $N^p = \Phi(N)$ to assume that N is abelian with trivial p-map.

Now the hypothesis on M implies that $[M,N]=0$. Thus M contains a central ideal of codimension 1, and hence is abelian.

The proof for $p=2$ is similar. Notice that if $N \triangleleft L$ then

$$N' \subseteq N^2 = \Phi(N) \triangleleft L.$$

Now write $N = M^2$. Again we have $N^2 + [N,L] < N$, so choose a maximal subalgebra N_1 of N which contains $N^2 + [N,L]$. By the induction hypothesis N_1 is powerful, so we may factor L by the ideal

$$\Phi(\Phi(N_1)) = (N_1^2)^2 = N_1^4$$

(cf. [RS, Proposition 5.2]). Then by the assumption on M we have $[N_1, M] \subseteq N_1^2$ and $[N_1^2, M] \subseteq (N_1^2)^2 = 0$. Thus in particular $[N_1, M^2] = 0$, so that N_1 is central in N. Now because $\dim N/N_1 = 1$, it follows that N is abelian, as required. ∎

Proposition 2.6: *If L does not involve W_n, then L^{p^n} is powerful if p is odd and $(L^{2^n})^2$ is powerful if $p=2$.*

Proof: By the previous lemma, it suffices to show that if $N \triangleleft L$ then

$$[N, L^{p^n}] \subseteq \Phi(N).$$

Suppose then, to the contrary, that there exists an ideal N of L such that L^{p^n} does not act trivially on $N/\Phi(N)$, and replace L by $L/\Phi(N)$ to assume that N is abelian and that the p-map acts trivially on N. Then there exist elements $x \in N$ and $y \in L$ such that $[x, y^{p^n}] \neq 0$. In particular, $[x,_{p^n-1} y] \neq 0$. Replacing x by $[x,_i y]$ for a suitable integer i we may assume that $[x,_{p^n-1} y] \neq 0$ but $[x,_{p^n} y] = 0$. Consider now the subalgebras $H = \langle x, y \rangle$ and $K = \langle y^{p^n} \rangle$. Then K is a central ideal of H and the section H/K of L is isomorphic to W_n, contradicting our hypothesis. ∎

Corollary 2.7: *If L does not involve W_n then L is $3p^n$-Engel if p is odd, and $3 \cdot 2^{n+1}$-Engel if $p=2$.*

Proof: First observe that every powerful Lie p-algebra I in $\mathcal{F}_p$ is nilpotent of class at most 2. Indeed, if $p > 2$ then

$$[I, I, I] \subseteq [I^p, I] \subseteq \gamma_{p+1}(I)$$

forces $\gamma_3(I) = 0$ by nilpotence. A similar argument works in the $p = 2$ case.

Next suppose again that $p > 2$. Then, according to Proposition 2.6, $I = L^{p^n}$ is powerful, and hence $\gamma_3(I) = 0$. Now let x, y be arbitrary elements of L. Then because I is an ideal of L, this yields

$$[x,_{3p^n} y] = [x, y^{p^n}, y^{p^n}, y^{p^n}] \in \gamma_3(I) = 0,$$

as required.

The proof for $p = 2$ is similar: because $\gamma_3((L^{2^n})^2) = 0$, we may deduce that $[x,_{3\cdot 2^{n+1}} y] = 0$. ∎

Finally, combining Corollaries 2.3 and 2.7 above we obtain the primary result of this section.

THEOREM 2.8: *If $L \in \mathcal{F}_p$ is n-power closed then L is $3p^{n+2}$-Engel if $p > 2$ and $3 \cdot 2^{n+3}$-Engel if $p = 2$.*

3. Proof of Theorem A

When we say that a Lie p-algebra L is residually nilpotent, we intend this to be interpreted in the category of Lie p-algebras.

To prove Theorem A, suppose that L is n-power closed and, for the sake of simplicity, that p is odd. Certainly it suffices to assume as well that L is 2-generated. Thus, in the case that L is nilpotent, $L/\zeta(L)$ lies in the class $\mathcal{F}_p$. Therefore, by Theorem 2.8, $L/\zeta(L)$ satisfies the $3p^{n+2}$-Engel condition, and hence L is $(3p^{n+2} + 1)$-Engel. Finally then, assume only that L is residually nilpotent. Then for every x, y in L we have

$$[x,_{3p^{n+2}+1} y] \in \bigcap_{i\geq 1} \gamma_i(L) = 0. \qquad \blacksquare$$

4. Proof of Theorem B

The implications (1)⇒(2)⇒(3) follow immediately from Lemma 1.1, while (1)⇒(4) follows directly from [R2, Theorem B]. To complete the proof of our Theorem B, it suffices to assume that R satisfies a polynomial identity which, firstly, implies n-power closure for some fixed n and, secondly, is a non-matrix identity (which is to say, it is not satisfied by the algebra of all (2×2)-matrices over the primary subfield of F). Consider now arbitrary elements x, y of R, and write S for the associative subalgebra of R generated by the finitely many nilpotent elements required to express x and y. Denote by $J(S)$ the Jacobson radical of S. Using standard PI-theory, we find that that $S/J(S)$ must be commutative because it satisfies a non-matrix identity. Consequently, $S/J(S)$ is nilpotent since it is generated by finitely many nilpotent elements. Using Braun's theorem [Br], it follows that S itself is nilpotent. Now let L be the restricted Lie subalgebra of S generated by x and y. It follows that L is finite dimensional with nilpotent p-map. But then, by Theorem A, L satisfies an Engel condition of degree bounded by a specific function of n and p only. Since x and y were arbitrary, the result follows. ∎

5. Proof of Theorem C

The associated Lie p-algebra (over $\mathbb{F}_p$) of an arbitrary group G is given by:

$$\mathcal{L}(G) = \bigoplus_{m \geq 1} D_m(G)/D_{m+1}(G).$$

Here $D_m(G)$ represents the mth dimension subgroup of G given by

$$D_m(G) = G \cap (1 + \Delta^n),$$

where Δ denotes the augmentation ideal of the group algebra $\mathbb{F}_p G$. The operations on $\mathcal{L}(G)$ are induced by commutation and exponentiation in G. See [L] for details.

Now suppose that G is a d-generated pro-p group. Then it is well-known that $\mathcal{L}(G)$ is d-generated and residually-$\mathcal{F}_p$: see [RS], for example. Therefore, by Corollary A2, $\mathcal{L}(G)$ is n-power closed for some n if and only if $\mathcal{L}(G)$ is nilpotent. But, as was proved in [L], this latter condition is equivalent to G being p-adic analytic. ∎

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